Asymptotic classes

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History and motivation

The study of asymptotic classes stems from a deep application by Chatzidakis, van den Dries and Macintyre (CDM) of the Lang–Weil estimates [6] and the work of Ax [1]:

Theorem (CDM). [2] *Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language of rings $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Then there exist a constant $C \in \mathbb{R}^{>0}$ and a finite set D of pairs $(d, \mu) \in \{0, \ldots, n\} \times \mathbb{Q}^{>0}$ such that for every finite field $\mathbb{F}_q$ and for every $\bar{a} \in {\mathbb{F}_q}^m$, if $\varphi({\mathbb{F}_q}^n, \bar{a}) \neq \varnothing$, then*

$$\big| |\varphi({\mathbb{F}_q}^n, \bar{a})| - \mu q^d \big| \leq C q^{d - 1/2} \qquad (*)$$

for some pair $(d, \mu) \in D$.

Furthermore, the parameters are definable; that is, for each $(d, \mu) \in D$ there exists an $\mathcal{L}_{\text{ring}}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathbb{F}_q$, $\mathbb{F}_q \models \varphi_{(d,\mu)}(\bar{a})$ iff $\bar{a}$ satisfies $(*)$ for (d, μ).

This theorem is a lot to take in at first. The idea is that the definable sets do not behave wildly and, even better, that we can in fact assign a dimension d and a measure μ to each of them in a finitary, uniform way.

Macpherson and Steinhorn investigated other classes of finite structures that satisfy the CDM theorem. [7] To this end they defined the notion of an *asymptotic class* as a generalisation of the CDM theorem. The definition given below is actually that given by Elwes in [4], which is itself a slight generalisation of the original definition in [7].

Definition. Let $\mathcal{L}$ be a first-order language, $N \in \mathbb{N}^+$ and $\mathcal{C}$ a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is an *N-dimensional asymptotic class* iff for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$,

(a) there exist a finite set $D \subset (\{0, \ldots, Nn\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ and a partition $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ of the set $\{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}$ such that for each $(d, \mu) \in D$

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \Big| = o\left(|M|^{d/N}\right)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$ as $|M| \to \infty$; and

(b) for each $(d, \mu) \in D$ there exists an $\mathcal{L}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models \varphi_{(d,\mu)}(\bar{a})$ iff $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$.

Remark. We distinguish between an $\mathcal{L}$-structure $\mathcal{M}$ and its underlying set M. The precise meaning of the o-notation is as follows: for every $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$, if $|M| > Q$, then

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \Big| < \varepsilon |M|^{d/N}.$$

Projection Lemma. [4, Lemma 2.2], [7, Theorem 2.1] *If the above definition holds for all formulae $\varphi(x, \bar{y})$ (with a single variable x), then it also holds for all formulae $\varphi(\bar{x}, \bar{y})$ (with a tuple of variables $\bar{x}$).*

Theorem. [4, Corollary 2.8], [7, Lemma 4.1] *Any infinite ultraproduct of an N-dimensional asymptotic class is supersimple of D-rank at most N.*

Examples

Some examples of 1-dimensional asymptotic classes include:

- The class of finite fields. [2]
- Families of finite difference fields $\{(\mathbb{F}_{p^{nk+m}}, \sigma^k) : k \in \mathbb{N}\}$, where p is prime, $m, n \in \mathbb{N}$ and σ is the Frobenius automorphism. [Ryten, PhD thesis; see [4, §4]]
- Various graph-theoretic examples. [7, Examples 3.3–3.6]
- The class of extraspecial groups of exponent p for a given fixed odd prime p. [7, Proposition 3.11] (A group of exponent p is *extraspecial* iff $G' = Z(G) = \Phi(G) \cong \mathbb{Z}/p\mathbb{Z}$, where $\Phi(G)$ is the Frattini subgroup of G, which is defined to be the intersection of all maximal subgroups of G or to be G if G has no maximal subgroups.)
- The class of finite cyclic groups. [7, Theorem 3.14]
- The collection of finite envelopes of any smoothly approximable linear, affine or projective geometry. [7, Theorem 3.8] (The notion of smooth approximation goes back to Lachlan and was developed in great depth by Cherlin and Hrushovski in [3]. We omit the definition; see [5, §4] for a concise description.)

Elwes expanded on this last example to show that for any smoothly approximable structure $\mathcal{M}$ there exists a subset of the set of finite envelopes of $\mathcal{M}$ that forms a $\mathrm{rk}(\mathcal{M})$-dimensional asymptotic class. [4, Proposition 4.1] Another example of an N-dimensional asymptotic class is any family of non-abelian finite simple groups of a fixed Lie rank, where N varies depending on the family. [Ryten, PhD thesis; see [5, Theorem 6.1]] These two examples were in fact the motivation for Elwes' generalisation of the original Macpherson–Steinhorn definition in [7], which covered only 1-dimensional asymptotic classes.

Further examples, results and exposition can be found in [4], [5], [7] and [8].

Multidimensional asymptotic classes

An N-dimensional asymptotic class consists of structures of a fixed dimension N. However, there is no a priori reason why CDM-like phenomena shouldn't occur in classes where the dimensions of the structures vary or where the structures themselves consist of different orthogonal/independent parts of different dimensions. With this thought in mind, Macpherson and Steinhorn have developed a further generalisation of an asymptotic class, a so-called *multidimensional asymptotic class* – or *mac* for short. The precise details of the definition of a mac have yet to be finalised, and Anscombe and I have come up with a variation of the Macpherson–Steinhorn definition that we feel captures CDM phenomena in broad generality (but isn't so general as to be trivial). We first need to cover some preliminaries before we state the definition:

Consider a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and define $\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}$.

Definition. Let $\{\Phi_i : i \in I\}$ be a partition of Φ. The set Φ_i is *definable* iff there exists an $\mathcal{L}$-formula $\psi(\bar{y})$ such that for every $\mathcal{M} \in \mathcal{C}$ and for every $\bar{a} \in M^m$, $(\mathcal{M}, \bar{a}) \in \Phi_i$ iff $\mathcal{M} \models \psi(\bar{a})$. The partition is *definable* iff Φ_i is definable for every $i \in I$. The partition is *finite* iff the indexing set I is finite.

Definition. Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. The class $\mathcal{C}$ is an *R-mac* in $\mathcal{L}$ iff for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist functions $h_1, \ldots, h_l \in R$ and a finite definable partition $\Phi_1, \ldots, \Phi_l$ of Φ such that for each $i \in \{1, \ldots, l\}$

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_i(\mathcal{M}) \Big| = o(h_i(\mathcal{M}))$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$ (with $\varphi(\mathcal{M}^n, \bar{a}) \neq \varnothing$) as $|M| \to \infty$.

The idea behind this definition is to extend the scope of the dimension–measure functions. In the case of an N-dimensional asymptotic class the functions are of the form $\mu |M|^{d/N}$: the only independent variable is the size of the whole structure. In an R-mac we have a lot more flexibility. Although the definition is very general and allows R to be *any* set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$, the motivating examples (see below) take R to be a set of functions defined in terms of the sizes of certain parts of the structures, such as sorts or equivalence classes. (Note that the original conception of a mac was in terms of sorts.) We can also vary the nature of the functions. In the old setting we have only multiplication by μ and exponents $\frac{d}{N}$ for $d \in \{0, \ldots, Nn\}$, but now we can be quite free and consider, for example, arbitrary rational or even irrational powers.

Some structural results about R-macs have already been obtained. For example, we have shown that the Projection Lemma also holds for R-macs, that no ultraproduct of an R-mac has the strict order property, and that if the set of functions R is isomorphic to the semi-ring $\mathbb{R}^{\geq 0}[X_1, \ldots, X_n]$, then any infinite ultraproduct of an R-mac is supersimple. The intended interpretations of these X_i are the sizes of certain parts of the structures (c.f. the previous paragraph), but the result still goes through if we view the semi-ring completely abstractly (modulo some technicalities). I am currently working on an adaptation to R-macs of a result of Elwes regarding N-dimensional asymptotic classes which, put roughly, states that bi-interpretability preserves being an asymptotic class. [4, §3]

Examples of R-macs

Some examples of R-macs are known:

- Every N-dimensional asymptotic class is an R-mac.
- In soon-to-be-submitted work, Darío García, Macpherson and Steinhorn have shown that the class of two-sorted structures consisting of a finite field F (in the language of rings) and a vector space V over F (in the language of additive groups with a map $F \times V \to V$) forms an R-mac, where the functions in R are rational polynomials in the sizes of the sorts. (They use slightly different language to state this, but it amounts to being a mac.) The ultraproduct of this class is supersimple. This class remains an R-mac if each structure is equipped with a bilinear form. However, the ultraproduct of this expanded class is no longer simple, although it is still NTP1.
- In the same manuscript, García, Macpherson and Steinhorn have shown that for any prime p, the class of groups $\{(\mathbb{Z}/p^n\mathbb{Z})^m : m, n > 0\}$ forms an R-mac. The details of functions in R are somewhat intricate and we will not state them here, but note that the functions do not fit the original conception of a mac and so this example lends credence to the generalisation to R-macs. Also note that this class is in fact an *exact multidimensional class*: the functions in R give the precise sizes of the definable sets, not just bounded approximations.

A non-example:

- The class of *rings* $\{(\mathbb{Z}/p^n\mathbb{Z})^m : m, n > 0\}$ is not an R-mac. This is because the formula $\varphi(x, y) :\equiv \exists z (z \cdot y = x)$ can pick out unboundedly many subsets of $\mathbb{Z}/p^n\mathbb{Z}$ of different sizes, since $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)| = p^{n-i}$ for $i \in \{0, \ldots, n\}$, and thus no *finite* set of functions $h_1, \ldots, h_l \in R$ is able to approximate the sizes of these definable sets for all n.

Questions

- We have found a sufficient condition on R for an infinite ultraproduct of an R-mac to be supersimple, but can we find a necessary condition?
- What conditions, if any, can we place on R to ensure that an infinite ultraproduct is NTP1?
- We conjecture that the set of *all* finite envelopes of *any* smoothly approximable structure forms an exact multidimensional class. Can we prove this?
- Can we find new, interesting examples of R-macs, especially ones that make full use of the generality of R? For instance, we hope to find graph-theoretic examples of bounded vertex-degree where the functions in R are defined in terms of sizes of $\varnothing$-definable sets. This would provide examples of R-macs not based on sorts, thereby adding further justification to the generalisation of the original conception of a mac to the current notion of an R-mac.

Measurable structures

So far we have covered only classes of finite structures, but an important infinite counterpart to asymptotic classes are so-called *measurable structures*. These are related to asymptotic classes in a fundamental way: any infinite ultraproduct of an asymptotic class is a measurable structure. [5, Proposition 3.9] As well as being interesting objects in their own right, measurable structures are invaluable for proving theorems about asymptotic classes: many of the proofs of the previously mentioned results go via measurable ultraproducts. Just as the older notion of an N-dimensional asymptotic class has been generalised to that of an R-mac, the concept of a measurable structure has been generalised to that of a *T-measurable structure*. We will cover only this more recent notion. Details of the original notion can be found in [5, §3] and [8, §5].

Definition. Let T be a semi-ring of characteristic zero and for an $\mathcal{L}$-structure $\mathcal{M}$ let $\mathrm{Def}(\mathcal{M})$ denote the set of all definable subsets of $\mathcal{M}^n$ for all $n > 0$. An infinite $\mathcal{L}$-structure $\mathcal{M}$ is *T-measurable* iff there exists a function $H : \mathrm{Def}(\mathcal{M}) \to T$ that satisfies the following conditions:

(a) For every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist $f_1, \ldots, f_l \in T$ such that for each $\bar{a} \in M^m$, $H(\varphi(\mathcal{M}^n, \bar{a})) = f_i$ for some $i \in \{1, \ldots, l\}$. Moreover, the set $\{\bar{a} \in M^m : H(\varphi(\mathcal{M}^n, \bar{a})) = f_i\}$ is $\varnothing$-definable for each f_i.

(b) $H(X) = |X|$ for all finite $X \in \mathrm{Def}(\mathcal{M})$ and $H(X_1 \cup \cdots \cup H_r) = H(X_1) + \cdots + H(X_r)$ for all disjoint $X_1, \ldots, X_r \in \mathrm{Def}(\mathcal{M})$.

(c) (Fubini) For all $X, Y \in \mathrm{Def}(\mathcal{M})$, if $\rho \colon X \to Y$ is a definable surjection and $H(\rho^{-1}(y)) = f$ for all $y \in Y$, then $H(X) = f \cdot H(Y)$.

As with N-dimensional asymptotic classes and measurable structures, any infinite ultraproduct of an R-mac is a T-measurable structure. Skipping over the details, the idea is that for a given $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ the functions $h_1, \ldots, h_l \in R$ give rise to the functions $f_1, \ldots, f_l \in T$. By moving to an infinite structure we can call upon the techniques and results of infinite model theory, bringing clear advantages.

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