UNIVERSITY OF LEEDS

Asymptotic classes and Lie coordinatisation

Daniel Wolf

Joint work with Sylvy Anscombe (UCLan),
Dugald Macpherson (Leeds) and Charles Steinhorn (Vassar)

Lancashire–Yorkshire Model Theory Seminar

University of Central Lancashire, 5th December 2015

Outline of the talk

- *R*-macs and *R*-mecs
- Lie coordinatisation
- Current work and conjectures

R-macs and *R*-mecs

We will skip the history of the development of the notion of an asymptotic class and dive straight into the latest definition. See [7] for some history and a survey of the current standing of the notion in the literature. The current work on *R*-macs appears in the manuscript [1] (work in progress).

Consider a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and let

$$\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}.$$

Definition (Definable partition)

Let $\{\Phi_i : i \in I\}$ be a partition of Φ. The set Φ_i is said to be *definable* iff there exists an $\mathcal{L}$-formula $\psi_i(\bar{y})$ with $l(\bar{y}) = m$ such that for every $\mathcal{M} \in \mathcal{C}$ and every $\bar{a} \in M^m$, $(\mathcal{M}, \bar{a}) \in \Phi_i$ iff $\mathcal{M} \models \psi_i(\bar{a})$. The partition is said to be *definable* iff Φ_i is definable for every $i \in I$ and to be *finite* iff the indexing set I is finite.

Notation: $\varphi(\mathcal{M}^n, \bar{a}) := \left\{\bar{b} \in \mathcal{M}^n : \mathcal{M} \models \varphi(\bar{b}, \bar{a})\right\}$

R-macs and *R*-mecs

Definition (*R*-mac)

Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. The class $\mathcal{C}$ is a *multidimensional asymptotic class for R in $\mathcal{L}$* iff for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist a finite definable partition $\Phi_1, \ldots, \Phi_k$ of Φ and functions $h_1, \ldots, h_k \in R$ such that for each $i \in \{1, \ldots, k\}$

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_i(\mathcal{M}) \Big| = o(h_i(\mathcal{M})) \tag{1}$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$ as $|M| \to \infty$.

For brevity we talk of an *R-mac in $\mathcal{L}$*.

R-macs and R-mecs

There is a stronger notion of a *multidimensional exact class for R in $\mathcal{L}$*, or *R-mec in $\mathcal{L}$* for short. This is where the previous definition holds, but instead of (1) we have equality, i.e. for each $i \in \{1, \ldots, k\}$

$$|\varphi(\mathcal{M}^n, \bar{a})| = h_i(\mathcal{M}) \tag{2}$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$.

Examples of *R*-macs

There are many! Here are a few:

- The class of all finite sets, where $\mathcal{L} = \varnothing$. This is in fact an *R*-mec.
- The class of finite cyclic groups, where $\mathcal{L} = \{+\}$. This is also an *R*-mec. [10, Theorem 3.14]
- The class of finite fields, where $\mathcal{L} = \{0, 1, +, \times\}$. [2]
- The class $\{(\mathbb{F}_{p^{nk+m}}, \sigma^k) : k \in \mathbb{N}\}$ of finite difference fields, where p is any prime, $n, m \in \mathbb{N}^+$ are coprime and $n > 1$, σ is the Frobenius automorphism ($\sigma(x) := x^p$) and $\mathcal{L} = \{0, 1, +, \times\}$.
  [Ryten, PhD thesis; see [6, §4]]
- Any family of non-abelian finite simple groups of a fixed Lie rank, e.g. $\mathrm{PSL}_n(\mathbb{F}_q)$, where n is fixed, q varies and $\mathcal{L} = \{\times\}$. [Ibid.]
- The class $\{(\mathbb{Z}/p^n\mathbb{Z})^m : n, m \in \mathbb{N}^+\}$ of **groups**, where p is any prime and $\mathcal{L} = \{+\}$. [8]

But there aren't *too* many...

Non-examples of *R*-macs

- The class $\mathcal{C}$ of all finite linear orders in (any expansion of) the language $\mathcal{L} = \{<\}$ does not form an *R*-mac for any *R*.

  Proof. Let $\varphi(x, y)$ be the formula $x < y$ and consider the finite total order $\mathcal{M}_n = \{a_0 < a_1 < \cdots < a_n\}$. Then $|\varphi(\mathcal{M}_n, a_i)| = i$. So as we let n increase and let i vary we define arbitrarily many subsets of **distinct** sizes. Thus no **finite** number of functions from R can approximate $|\varphi(\mathcal{M}_n, a_i)|$ for all $n, i \in \mathbb{N}$. □

- Let p be prime. Then the class $\{(\mathbb{Z}/p^n\mathbb{Z})^m : n, m \in \mathbb{N}^+\}$ of **rings** in (any expansion of) the language $\mathcal{L} = \{+, \times\}$ does not form an *R*-mac for any *R*.

  Proof. Let $\varphi(x, y)$ be the formula $\exists z\, (x = z \times y)$. Then $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)| = p^{n-i}$. So as we let n increase and let i vary we define arbitrarily many subsets of distinct sizes and thus no finite number of functions from R can approximate $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)|$ for all $n, i \in \mathbb{N}$. (Notice that we didn't need to consider m.) □

Smooth approximation

In order to understand Lie coordinatisation it is helpful to first consider smooth approximation, a notion invented by Lachlan the 1980s and then further developed by him and others, e.g. [3, 4, 9].

Definition (Smooth approximation)

Let $\mathcal{N}$ be a finite $\mathcal{L}$-substructure of an $\aleph_0$-categorical $\mathcal{L}$-structure $\mathcal{M}$. $\mathcal{N}$ is a *homogeneous substructure* of $\mathcal{M}$ iff for every $k > 0$ and every pair $\bar{a}, \bar{b} \in \mathcal{N}^k$, $\bar{a}$ and $\bar{b}$ lie in the same Aut($\mathcal{M}$)-orbit iff they lie in the same $\mathrm{Aut}_{\{\mathcal{N}\}}(\mathcal{M})$-orbit, where $\mathrm{Aut}_{\{\mathcal{N}\}}(\mathcal{M})$ is the set of autmorphisms of $\mathcal{M}$ that fix $\mathcal{N}$ setwise.

An $\aleph_0$-categorical structure $\mathcal{M}$ is *smoothly approximated* iff there exist $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$ such that each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

Examples of smoothly approximated structures

(1) Trivial example: Let $\mathcal{M}$ be a countably infinite set in the language of equality. Enumerate $\mathcal{M}$ by $(m_i : i < \omega)$ and let $\mathcal{M}_i = \{m_0, \ldots, m_i\}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

(2) Let $\mathcal{M}$ be the unique countable structure consisting of infinitely many E_1-equivalence classes and a refinement E_2 such that each E_2-equivalence class is also infinite, i.e. first partition $\mathcal{M}$ into infinitely many E_1-classes and then partition each E_1-class into infinitely many infinite E_2-classes. Enumerate the E_1-classes by $(e_j : j < \omega)$ and the E_2-classes within each e_j by $(e_{jk} : k < \omega)$. Finally, enumerate the elements of each e_{jk} by $(e_{jkn} : n < \omega)$. Let $\mathcal{M}_i := \{e_{jkn} : j, k, n \leq i\}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

(3) Let $\mathcal{M}$ be the direct sum of ω-many copies of $\mathbb{Z}/p^2\mathbb{Z}$ (notation: $\mathcal{M} = (\mathbb{Z}/p^2\mathbb{Z})^\omega$), where p is some fixed prime. Let $\mathcal{M}_i$ consist of the first i copies of $\mathbb{Z}/p^2\mathbb{Z}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

Lie coordinatisation

Lie coordinatisation is ostensibly a different notion from that of smooth approximation, but they turn out to be equivalent. [4, Theorem 2] The definition of Lie coordinatisation is complicated, but its advantage over smooth approximation is that it gives us much more information about the internal structure of the structure. Its full definition is too long to give here (see [4, Definition 2.1.10] and [5, Definition 1.3]), but we will sketch it, omitting various details, and then go over two examples, which should hopefully bring some intuition.

We start with a finite list of *linear Lie geometries*, which are certain kinds of spaces over a finite field K, namely: a pure set; a pure vector space; a polar space, a union of two vector spaces V and W a bilinear form $V \times W \to K$; an inner product space, a vector space with a sesquilinear form; an orthogonal space, a vector space with a quadratic form; and a quadratic geometry, complicated. One then has projective and affine versions of these linear geometries.

Lie coordinatisation

Sketch definition (Lie coordinatisation)

An $\aleph_0$-cetegorical structure $\mathcal{M}$ is *Lie coordinatised* iff $\mathcal{M}$ admits a tree structure in the following way: For each $a \in \mathcal{M}$ there exists a finite sequence $a_0 < \cdots < a_n = a$ in $\mathcal{M}$ with $a_{i-1} \in \mathrm{dcl}(a_i)$ such that for each i either:

(i) $a_i \in \mathrm{acl}(a_{i-1})$;

(ii) for some $j < i$ there exists an a_j-definable projective Lie geometry J with $a_i \in J$; or

(iii) for some $j < k < i$ there exists an a_j-definable projective Lie geometry J and an a_k-definable affine Lie geometry (V, A) with vector part V and $a_i \in A$ such that the projectivisation of V is J.

We also specify that there is a $\varnothing$-definable root, i.e. some $c \in \mathcal{M}$ (or possibly $c = \varnothing$) such that $c \leq a$ for all $a \in \mathcal{M}$.

A structure is *Lie coordinatisable* iff it is bi-interpretable with a Lie coordinatised structure.

Examples of Lie coordinatisable structures

(1) Recall the earlier example of the countable structure $\mathcal{M}$ with two equivalence relations E_1 and E_2. We will show that it is Lie coordinatis**able**, rather than Lie coordinat**ised**, by showing that the three-sorted structure $\mathcal{M} \cup \mathcal{M}/E_1 \cup \mathcal{M}/E_2$ is Lie coordinatised. At the root of the tree we place $\varnothing$. Above the root we place the E_1-classes. Above them we place the E_2-classes, with each E_2-class directly above the E_1-class in which it is contained. Finally, we place $\mathcal{M}$ above the E_2-classes, with each $a \in \mathcal{M}$ directly above a/E_2.

Examples of Lie coordinatisable structures

(2) [4, Example 2.1.11] Recall the earlier example of the direct sum $\mathcal{M} := (\mathbb{Z}/p^2\mathbb{Z})^\omega$. Let $\mathcal{M}[p] := \{x \in \mathcal{M} : px = 0\}$. So $\mathcal{M}[p] = (p\mathbb{Z}/p^2\mathbb{Z})^\omega \cong (\mathbb{Z}/p\mathbb{Z})^\omega$. At the root we place 0. Above the root we place the projectivisation of $\mathcal{M}[p]$, which we will denote by $P(\mathcal{M}[p])$. (The projectivisation is given by quotienting out by the relation $\mathrm{acl}(x) = \mathrm{acl}(y)$.) Above that we place $\mathcal{M}[p] \setminus \{0\}$, which covers each point in $P(\mathcal{M}[p])$ by the corresponding finite set of points above it. Finally, we place $\mathcal{M} \setminus \mathcal{M}[p]$ above $\mathcal{M}[p] \setminus \{0\}$, which covers each $a \in \mathcal{M}[p] \setminus \{0\}$ by the affine space $\mathcal{M}_a := \{x \in \mathcal{M} : px = a\}$. Again notice that we have shown that the structure $\mathcal{M}$ is Lie coordinatisable, rather than Lie coordinatised, as we had to add the elements of $P(\mathcal{M}[p])$, which are not elements of the original structure $\mathcal{M}$.

Envelopes

The definition of an envelope of a Lie coordinatised structure is fairly involved (see [4, Definition 3.1.1]), but the intuition is much less so. Envelopes are examples of finite homogeneous structures that smoothly approximate the structure. For example, in all the examples of smoothly approximated structures given earlier, the $\mathcal{M}_i$ are envelopes.

It is through envelopes that the link between smooth approximation/Lie coordinatisation and asymptotic classes comes in.

Current work

I am currently in the midst of proving the following:

Proposition (almost...)

Let $\mathcal{M}$ be a Lie coordinatisable structure and let $\mathcal{E}$ be a collection of envelopes for $\mathcal{M}$. Then $\mathcal{E}$ forms an R-mec.

(For simplicity I have omitted some technical conditions.)

This should follow from the proof of [4, Proposition 5.2.2]; a similar result was proved by Elwes [6, Proposition 4.1]. The novelty lies in translating the work from the Lie coordinatisation context to the R-mec context, the details being somewhat involved.

Conjectures

Macpherson has conjectured the following two statements:

Precise conjecture

Fix some language $\mathcal{L}$ and some $d \in \mathbb{N}^{+}$ and let $\mathcal{C}_{\mathcal{L},d}$ be the class of all finite $\mathcal{L}$-structures with at most d 4-types. Then $\mathcal{C}_{\mathcal{L},d}$ is an R-mec for some R.

Less precise conjecture

A structure coordinatised by an R-mac/-mec is an R'-mac/-mec, where R' is some set of functions derived from R.

The first conjecture seems quite likely and it should follow from the results obtained by Cherlin and Hrushovski in [4] and the statement on the previous slide. The second conjecture is less certain; for example, what exactly should it mean to be coordinatised by an R-mac? We hope to find out!

Thank you for your attention!

Slides available at:

www.dwolf.eu/research

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